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НАУКА И ОБРАЗОВАНИЕ ЗА
ИНТЕЛИГЕНТЕН РАСТЕЖ

BG05M2OP001-1.001-0003
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Investigations on some polynomial Lienard–type systems: number of limit cycles, simulations

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ИКТ

Проект BG05M2OP001-1.001-0003 „Център за върхови постижения по Информатика и информационни и комуникационни технологии“, финансиран от Оперативна програма „Наука и образование за интелигентен растеж“, съфинансирана от Европейския съюз чрез Европейските структурни и инвестиционни фондове.

We consider a new extended Lienard–type planar system with the polynomial P_{2n+1} of the best Hausdorff approximation of the function $sgn\,(x)$. The number of limit cycles is also studied. The discussions are related to solving some technical problems such as the synthesis of antennas and electrical circuits.

In 1984, Blows and Lloyd [3] proved the following interesting theorem concerning the number of limit cycles of the Lienard system [4]

$$\begin{cases} \frac{dx}{dt} = y - \varepsilon F(x) \\ \frac{dy}{dt} = -x \end{cases} \quad (1)$$

Theorem (Blows and Lloyd). The Lienard system with

$$F(x) = a_1 x + a_2 x^2 + \cdots + a_{2m+1} x^{2m+1}$$

has at most m local limit cycles and there are coefficients with $a_1 . a_3, \ldots, a_{2m+1}$ alternating in sign such that Lienard system has m local limit cycles.

In [5] the authors consider a general slow–fast Lienard system

$$\begin{cases} x' = y - F(x) \\ y' = -\varepsilon(x - \lambda) \end{cases}$$

F is a Morse function. The authors prove that for a well-chosen polynomial f of degree 6, the equation $x'' + f(x)x' + x = 0$ exhibits 4 limit cycles. It induces that for $n \geq 3$ there exists polynomial of degree $2n$ such that the related equations exhibit more than n limit cycles. This contradicts the conjecture of Lins, de Melo and Pugh starting that for Lienard equations as above, with f of degree $2n$, the maximum number of limit cycles is n.

The Melnikov function [6] for the system

$$\begin{cases} \dfrac{dx}{dt} & = & y - \varepsilon(a_1 x + a_2 x^2 + \cdots + a_{2n+1} x^{2n+1}) \\ \dfrac{dy}{dt} & = & -x \end{cases} \qquad (2)$$

is defined as

$$M(\alpha, \mu) = -2\pi\alpha^2 \left(\frac{a_1}{2} + \frac{3}{8} a_3 \alpha^2 + \cdots + \binom{2n+2}{n+1} \frac{a_{2n+1}}{2^{2n+2}} \alpha^{2n} \right) \quad (3)$$

The *Melnikov polynomial* is defined as

$$P(r^2, n) = -\frac{1}{2\pi r^2} M(r, \mu). \qquad (4)$$

The following result due to Perko and co–workers [7]–[8] provides the necessary information about the number of limit cycles and their radii

Theorem [7]–[8]. The Lienard system (2) for sufficiently small $\varepsilon \neq 0$ has at most n limit cycles asymptotic to circles of radii $r_j, j = 1,2, \ldots, n$ as $\varepsilon \to 0$ if and only if the nth degree polynomial in r^2,

$$P(r^2, n) = \frac{a_1}{2} + \frac{3}{8} a_3 r^2 + \cdots + \binom{2n+2}{n+1} \frac{a_{2n+1}}{2^{2n+2}} r^{2n} \quad (5)$$

has n positive roots $r^2 = r_j^2, j = 1,2, \ldots, n$.

Denote by P_{2n+1} the polynomial of the best Hausdorff approximation of the function $sgn\,(x)$ in $[-1,1]$ by algebraic polynomials of degree $2n+1$ [1]. The polynomials take part in some technical problems like antenna synthesis and electrical schemes [2].

We consider a new extended Lienard–type planar system with the polynomial P_{2n+1}. The number of limit cycles is also studied.

Main Results. Simulations

Extended Lienard–type planar system

We consider the following model of the type:

$$\begin{cases} \frac{dx}{dt} & = & y - \varepsilon P_{2n+1}(x) \\ \frac{dy}{dt} & = & -x \end{cases} \quad (6)$$

where $\varepsilon > 0$ and $P_{2n+1}(x)$ is the polynomial of the best Hausdorff approximation of the function $sgn\,(x)$. For example for $n = 3$ and $n = 6$ we have

$$P_7(x) \quad = \quad 5.40305164x - 20.50556646x^3 + 32.39192287x^5 - 16.461x^7$$

$$P_{13}(x) \quad = \quad 8.867176x - 104.8702x^3 + 629.1424x^5 - 1864.313x^7 + 2845.135x^9 - 2145.837x^{11} + 632.9946x^{13}$$

(see Fig. 1).

Under certain conditions it can be shown that Lienard's–type system (6) has a limit cycle. The proof of existence of limit cycle is based on the verification of the conditions in Lienard's theorem [4] and we will omit it here.

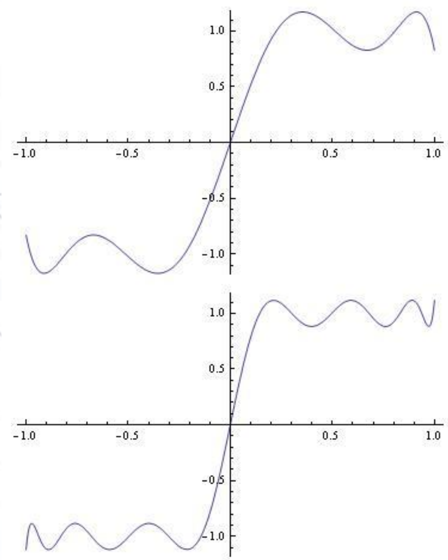

Figure 1: The polynomials $P_{2n+1}(x)$ for $n = 3$ and $n = 6$.

The simulation for user-selected coefficient $\varepsilon = 0.006$ and $n = 3$, with the model (6) is shown in Fig. 2.

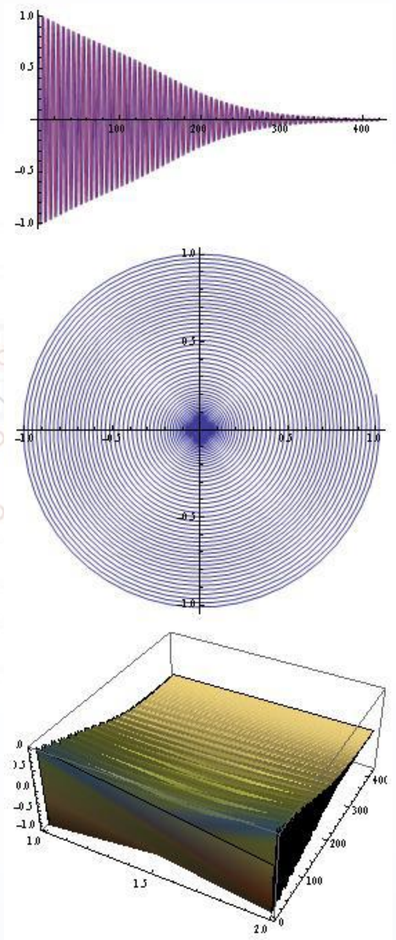

Figure 2: The solutions of the planar system for $\varepsilon = 0.006$ and $n = 3$.

The new model in the light of Melnikov's considerations

The case $n = 3$.

Consider the model

$$\begin{cases} \frac{dx}{dt} &= y - \varepsilon(\mu x - 20.50556646x^3 + 32.39192287x^5 - 16.461x^7) \\ \frac{dy}{dt} &= -x \end{cases} \qquad (7)$$

where $\mu > 0$, $\varepsilon > 0$.

The following is valid

Theorem. The Lienard–type system (7) for $n = 3$, and for all sufficiently small $\varepsilon \neq 0$

a) for $0 < \mu < 0.00487909525\ldots$ has three hyperbolic limit cycles with radii r_1, r_2 and r_3.

b) for $\mu = 0.00487909525$ has a simple limit cycle and limit cycle with multiplicity – two.

Proof. For the Melnikov polynomial in r^2 (see Fig. 3) we have:

$$P(r^2, 3) = \frac{\mu}{2} - 0.31319580r^2 + 10.12247589r^4 - 4.50105468r^6. \qquad (8)$$

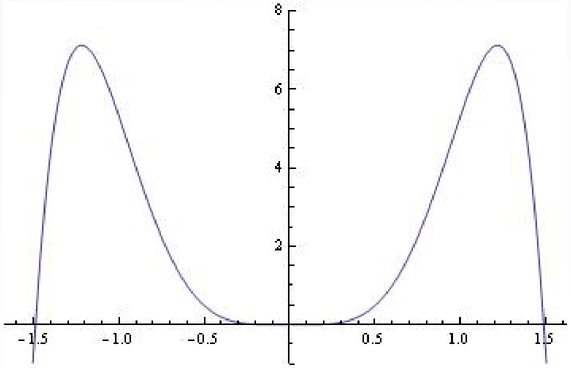

Figure 3: The Melnikov polynomial $P(r^2, 3)$ for $n = 3$ and $\mu = 0.00487909525$. The roots are: 1.48918 - simple and 0.12503 - with multiplicity two.

Some values of the radii r_i; $i = 1,2,3$ as functions of μ in the interval $0 < \mu < 0.00487909525$ are tabulated in Table 1.

μ	r_1	r_2	r_3
0.001	0.0410917	0.17224	1.489151
0.002	0.0601203	0.166488	1.48915
0.003	0.0769316	0.159346	1.48916
0.00487909525	0.12503	0.125037	1.48918

Table 1: The values of μ that result in a three positive roots of the Melnikov polynomial (Theorem 1); for $\mu = 0.00487909525$: a simple limit cycle and limit cycle with multiplicity – two.

For an appropriate choice of the parameter μ (for example $\mu = 0.245602716$) we have a limit cycle and limit cycle with multiplicity two (see Table 1).

This completes the proof of the theorem.

The case $n = 6$.

The extended Van der Pol equation [9] corresponding to planar system is of the form:

$$x'' + \varepsilon P'_{13}(x)x' + x = 0. \quad (9)$$

The solution of the equation (9) for user-selected coefficient $\varepsilon = 0.006$, $x(0) = 1$, $x'(0) = 0.1$ is depicted in Fig. 4.

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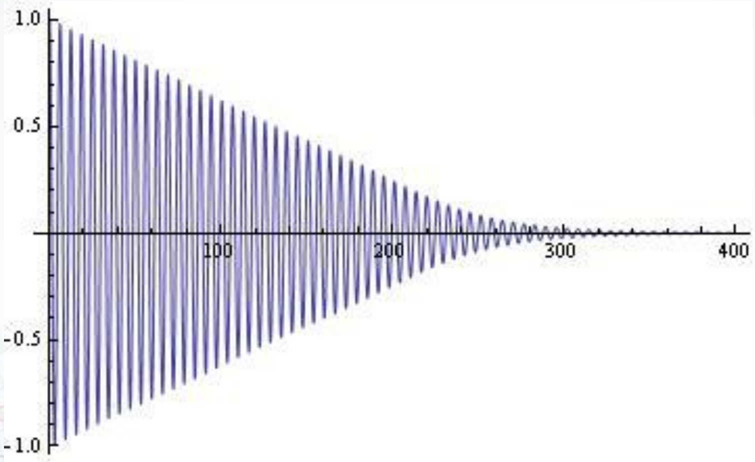

Figure 4: The solution of the equation (9) for user-selected coefficient $\varepsilon = 0.006$, $x(0) = 1$, $x'(0) = 0.1$.

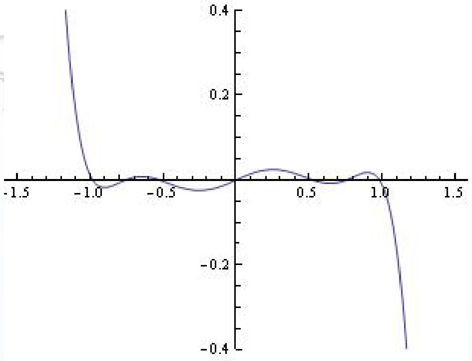

Figure 5: The polynomial $g(x)$.

Consider a Lienard system of type

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = g(x) + \varepsilon f(x) y \end{cases} \qquad (10)$$

where $0 \leq \varepsilon \leq 1$ and $g(x) = 0.15169x - 0.97264x^3 + 1.80902x^5 - x^7$ (see Fig. 5), $f(x) = P_7(x) = 5.40305x - 20.50556646x^3 + 32.39192287x^5 - 16.461x^7$

The solution of the system (10) for $\varepsilon = 0.001$ is visualized on Fig. 6.

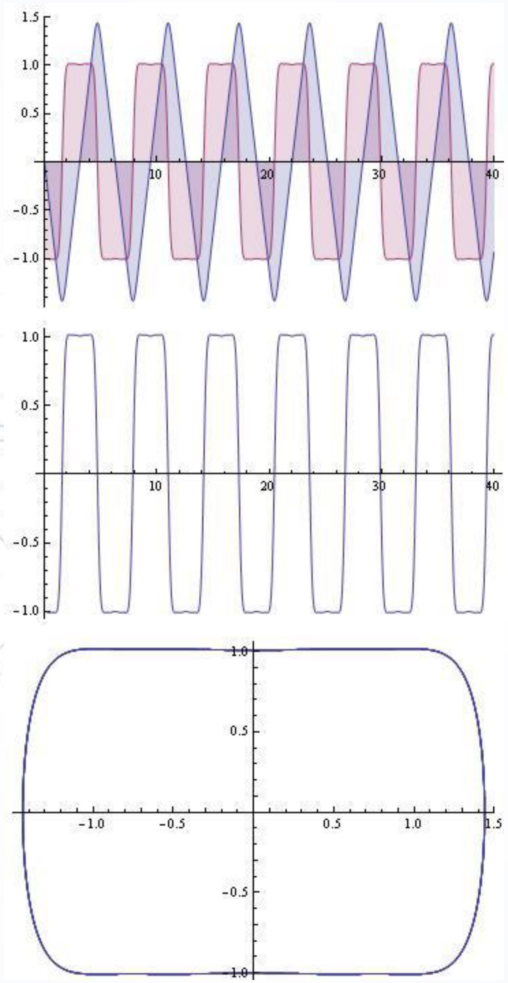

Figure 6: a) The solutions of the planar system (10) for $\varepsilon = 0.001$; b) the y-component of the solution; c) the portrait of the planar system.

Concluding remarks

The discussions are related to solving some technical problems such as the synthesis of antennas and electrical circuits. Other interesting radiation diagrams can be generated with an appropriate choice of the functions $f(x)$ and $g(x)$ appearing in the Lienard system. For other results see [10]–[12].

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